

3-Colourability of Dually Chordal Graphs in Linear Time

Arne Leitert

University of Rostock, Germany
arne.leitert@uni-rostock.de

Abstract. A graph G is dually chordal if there is a spanning tree T of G such that any maximal clique of G induces a subtree in T . This paper investigates the Colourability problem on dually chordal graphs. It will show that it is NP-complete in case of four colours and solvable in linear time with a simple algorithm in case of three colours. In addition, it will be shown that a dually chordal graph is 3-colourable if and only if it is perfect and has no clique of size four.

1 Introduction

Colouring a graph is the problem of finding the minimal number of colours required to assign each vertex a colour such that no adjacent vertices have the same colour. It is a classic problem in computer science and one of Karp's 21 NP-complete problems [8]. Colouring remains NP-complete if bounded to three colours [11]. Also, if a 3-colourable graph is given, it remains NP-hard to find a colouring, even if four colours are allowed [7].

Dually chordal graphs are closely related to chordal graphs, hypertrees and α -acyclic hypergraphs, which also explains the name of this graph class (see [2] for more). Section 3 will show, that Colouring is NP-complete for dually chordal graphs if bounded to four colours. So the aim of this paper is to find a linear time algorithm for the 3-Colouring problem.

Next to the classic vertex colouring problem, there are the problems of colouring only edges or edges and vertices of a graph. For dually chordal graphs they have been investigated by de Figueiredo et al. [5].

2 Basic Notions

Let $G = (V, E)$ be a graph with the vertex set V , the edge set E . For this paper, any graph is finite, undirected, connected and without loops or multiple edges. For a set $U \subseteq V$, $G[U]$ denotes the induced subgraph of G with the vertex set U . Let $\bar{G} = (V, \bar{E})$ be the complement of G , such that $\bar{E} = \{uv \mid u, v \in V; u \neq v; uv \notin E\}$.

A set of vertices $S \subseteq V$ is a *clique* if for each pair $u, v \in S$ ($u \neq v$), u and v are adjacent. A clique of size i is denoted as K_i . The number of vertices in a maximum clique in G is the *clique number* $\omega(G)$ of G . Let a *diamond* be a

graph with four vertices and five edges. The edge connecting the vertices with degree 3 is called *mid-edge*. A *chordless cycle* C_k has k vertices v_1, \dots, v_k and the edges $v_i v_{i+1}$ (index arithmetic modulo k). If a chordless cycle has at least five vertices, it is also called *hole*. A *wheel* W_k is a C_k plus a vertex v_w and the edges $v_w v_i$ ($1 \leq i \leq k$). A chordless cycle C_k or wheel W_k is *odd* if k is odd.

Let $N(v) := \{u \in V \mid uv \in E\}$ denote the *open* and $N[v] := N(v) \cup \{v\}$ the *closed neighbourhood* of the vertex v . A graph is *locally connected* if for all vertices v the open neighbourhood $N(v)$ is connected.

A vertex $u \in N[v]$ is a *maximum neighbour* of v , if for all $w \in N[v]$, $N[w] \subseteq N[u]$ holds. Note that $u = v$ is not excluded. A vertex ordering (v_1, \dots, v_n) is a *maximum neighbourhood ordering* if every v_i ($1 \leq i \leq n$) has a maximum neighbour in $G[\{v_i, \dots, v_n\}]$.

A vertex v is an articulation point of G if $G[V \setminus \{v\}]$ is not connected. If a graph has no articulation point it is *biconnected* (or *2-connected*). Maximal biconnected subgraphs are called *blocks*. Any connected graph can be decomposed into a block tree. If two blocks intersect, they have exactly one common vertex which is an articulation point.

An *independent set* is a vertex set $I \subseteq V$ such that for all vertices $u, v \in I$, $uv \notin E$ holds. A graph is *k-colourable* if V can be partitioned into k independent sets V_1, \dots, V_k with $V = V_1 \cup \dots \cup V_k$ and $V_i \cap V_j = \emptyset$ ($i \neq j$). The *chromatic number* $\chi(G)$ of a graph G is the lowest k such that G is k -colourable. The *k-Colourability* problem asks if a graph is k -colourable. Accordingly the *Colourability* problem asks for the chromatic number of a graph. It is easy to see, that 2-Colourability can be solved in linear time for each graph.

A graph G is *perfect* if for each induced subgraph G^* of G the chromatic and the clique number are equal ($\omega(G^*) = \chi(G^*)$). Chudnovsky et al. [4] have shown that a graph is perfect if and only if it is $(C_{2n+5}, \overline{C_{2n+5}})$ -free for all $n \geq 0$. This is known as the Strong Perfect Graph Theorem. Perfect graphs can be recognised and coloured in polynomial time [3][6].

For a graph G , $K(G)$ denotes the *clique graph* of G , where each vertex in $K(G)$ represents a maximal clique of G and two vertices are connected if the corresponding cliques have a common vertex, i.e. $K(G) = (V, E)$ with $V = \{k \mid k \text{ is a maximal clique of } G\}$ and $E = \{k_1 k_2 \mid k_1 \cap k_2 \neq \emptyset\}$. A graph G is *chordal* if it is C_k -free ($k \geq 4$). A graph is *clique-chordal* if its clique graph is chordal.

3 Dually Chordal Graphs

Dually chordal graphs are originally defined by a maximum neighbourhood ordering.

Definition 1. *A graph is dually chordal if and only if it has a maximum neighbourhood ordering.*

Brandstädt et al. [2] give an overview of characterisations for dually chordal graphs. One of it allows to recognise graphs of this class in linear time (with an

algorithm by Tarjan and Yannakakis [12]). Another characterisation was developed in [10]:

Lemma 1 ([2], [10]). *Let $P_T(u, v)$ the set of vertices on the path from u to v in the tree T , with $u, v \notin P_T(u, v)$. For a graph $G = (V, E)$ the following conditions are equivalent:*

1. *G is dually chordal.*
2. *There is a spanning tree T of G such that every maximal clique of G induces a subtree in T .*
3. *There is a spanning tree T of G such that for every edge $uv \in E$ the following holds: $\forall w \in P_T(u, v) : uw, vw \in E$*

Lemma 2 ([10]). *The spanning trees in conditions 2 and 3 of Lemma 1 are equal.*

Since $P_T(u, v)$ does not include the vertices u and v , (according to the notation for the open and closed neighbourhood) let $P_T[u, v] := P_T(u, v) \cup \{u, v\}$.

Corollary 1. *If $uv \in E$ then $P_T[u, v]$ is a clique.*

It is easy to see that an arbitrary graph becomes dually chordal if a vertex adjacent to all vertices is added. This leads to a reduction for several NP-complete problems, including Colourability. The idea for this method was already given by Brandstädt et al. [1].

Theorem 1. *4-Colourability is NP-complete for dually chordal graphs.*

Proof. Let $G = (V, E)$ be an arbitrary graph. Thus, $G_{dc} = (V \cup \{u\}, E \cup \{uv \mid v \in V\})$ with $u \notin V$ is dually chordal. It follows that G is 3-colourable if and only if G_{dc} is 4-colourable. Because 3-Colourability is NP-complete in general [11], 4-Colourability is NP-complete for dually chordal graphs. \square

4 K_4 -free Dually Chordal Graphs

Obviously a graph has to be K_4 -free to be 3-colourable. So this section investigates K_4 -free dually chordal graphs.

For this section let $G = (V, E)$ be a K_4 -free dually chordal graph. The spanning tree $T = (V, E_t)$ of G and the set $P_T[u, v]$ are defined as in condition 3 of Theorem 1 and the paragraph after Lemma 2.

Corollary 2. *If $uv \in E$ then $|P_T[u, v]| \leq 3$, i.e. $uv \in E \wedge uv \notin E_t \Rightarrow \exists! w : uw, vw \in E_t$.*

For the next lemma the notion of a *route* is defined:

Definition 2. *A route on G is a list of vertices (v_1, \dots, v_k) such that $v_i v_{i+1} \in E$ or $v_i = v_{i+1}$ for all $i < k$.*

Lemma 3. *Let \mathcal{C} be a C_k in G with $k \geq 4$. There is a vertex w such that \mathcal{C} and w form a wheel. Also no edge of \mathcal{C} is in T .*

Proof. Let \mathcal{C} be a chordless cycle with the vertices $\{c_1, \dots, c_k\}$ ($k \geq 4$) and the edges $c_i c_{i+1}$ (index arithmetic modulo k). Based on Corollary 2 for every edge there is the set of vertices $P_T[c_i, c_{i+1}] = \{c_i, w_i, c_{i+1}\}$ with $c_i = w_i$ if and only if $c_i c_{i+1} \in E_t$. Thus, it is possible to build a route $\rho = (c_1, w_1, c_2, \dots, w_k, c_1)$ on T (see Figure 1).

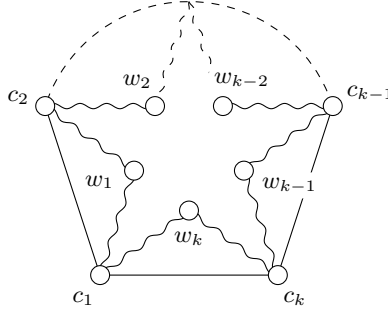


Fig. 1. The circle \mathcal{C} and the route ρ (waved).

Because all the edges of ρ are in T , ρ also induces a tree. Therefore ρ can be eliminated by removing leaves. Let $(\dots, v_{i-1}, v_i, v_{i+1}, \dots)$ be a subroute of ρ and v_i a leaf. Thus, if $v_{i-1} \neq v_i$ and $v_i \neq v_{i+1}$ then $v_{i-1} = v_{i+1}$, so v_{i-1} or v_{i+1} can also be removed from ρ with v_i .

Now assume that no vertex of \mathcal{C} is a leaf. In this case a vertex w_i is a leaf. So it follows, that $c_i = c_{i+1}$. Therefore there must be a leaf c_i and also $w_{i-1} = w_i$ (otherwise $c_{i-1}c_{i+1} \in E$). This allows to remove c_i and w_i from ρ .

Continuing this procedure, it follows that $w_1 = \dots = w_k$ and $c_i \neq w_i$ for all i . Thus, no edge $c_i c_{i+1}$ is an edge of T and there is a vertex w , such that \mathcal{C} and w form a wheel. \square

Theorem 2. *A dually chordal graph G is 3-colourable if and only if G is perfect and K_4 -free.*

Proof. \Leftarrow : By definition of perfect graphs.

\Rightarrow : G is 3-colourable, so it is K_4 -free. Let $k \geq 7$. It is easy to see, that no $\overline{C_k}$ is 3-colourable. Therefore, G has no $\overline{C_k}$. From the 3-colourability it also follows that there is no odd wheel in G and by Lemma 3 no odd hole¹. Thus, by the Strong Perfect Graph Theorem, G is perfect (recall, $C_5 = \overline{C_5}$). \square

Theorem 2 now leads to a polynomial time algorithm for 3-Colourability of dually chordal graphs.

¹ also no $\overline{C_6}$

5 Linear Time Algorithm

After showing that there is a polynomial time algorithm, this section presents a linear time algorithm for 3-Colourability of dually chordal graphs based on the following theorem:

Theorem 3. *Each connected locally connected graph can be constructed by starting with an arbitrary edge and then add only vertices that has at least two adjacent neighbours in the already constructed graph.*

Proof. Let (u, v, v_1, \dots, v_k) be the ordering the vertices are added to construct a locally connected graph $G = (V, E)$ starting with an edge uv . Also let $V_i := \{u, v, v_1, \dots, v_i\}$ with $V_0 := \{u, v\}$ and $0 \leq i \leq k$.

Now let $G_i := G[V_i]$ ($0 \leq i < k$) be a locally connected subgraph of G . Assume, there is no vertex w adjacent to two vertices in G_i . Then G is not connected or w has only one neighbour v_n in G_i . In second case, because G is locally connected, there is a vertex set $S \subseteq N(v_n) \setminus V_i$ connecting w with the neighbours of v_n that are already in G_i . Thus, there is a vertex $s \in S$ adjacent to v_n and a neighbour of v_n in G_i . Therefore, set $v_{i+1} := s$. \square

Theorem 3 allows to colour graphs whose blocks are locally connected by using the following strategy: Select an uncoloured vertex with a minimal number of available colours and give it an available one. Repeat this until all vertices are coloured or there is no available colour for a vertex. Algorithm 1 describes this more detailed:

Algorithm 1.

Input: A graph $G = (V, E)$ whose blocks are locally connected.

Output: A 3-colouring for G if and only if G is 3-colourable.

1. Give every vertex v a set of available colours $c(v) := \{1, 2, 3\}$. Also every vertex gets the possibility to get marked as coloured. At the beginning no vertex is marked.
2. **While** There are uncoloured vertices
 - (a) Select an uncoloured vertex v for which $|c(v)|$ is minimal.
 - (b) If $c(v) = \emptyset$, **STOP:** G is not 3-colourable.
 - (c) Disable all but one colour in $c(v)$ and mark v as coloured.
 - (d) Disable $c(v)$ for all neighbours of v ($\forall u \in N(v) : c(u) := c(u) \setminus c(v)$).

Theorem 4. *Algorithm 1 works correctly and can be implemented in linear time.*

Proof.

Correctness. After selecting the first two vertices, the algorithm only selects vertices with two adjacent neighbours, i.e. with $|c(v)| \leq 1$, until all vertices of a locally connected block are coloured (Theorem 3). Therefore, the colouring is unique for a locally connected block.

Also, if a vertex v with $|c(v)| = 2$ and the coloured neighbour u is selected, u is an articulation point. This allows to treat u and v as the first coloured vertices of a locally connected graph. Therefore, each available colour for v leads to a correct colouring.

Complexity. It is easy to see, that line 1 runs in linear time and line 2(b) in constant time such as line 2(c). Line 2(d) is bounded by the number of neighbours. Because $\sum_{v \in V} |N(v)| = 2|E|$, the full time needed (overall iterations) for line 2(d) is in $\mathcal{O}(|E|)$.

Line 2(a) can be implemented by using three doubly linked lists as queues. The lists include the uncoloured vertices. One is for vertices with $|c(v)| = 3$, one for $|c(v)| = 2$ and one for $|c(v)| \leq 1$. If there is also a pointer from each vertex to its entry in the list, adding and removing a vertex can be done in constant time. Thus, line 2(a) runs in constant time. \square

Another linear time algorithm for 3-Colourability of locally connected graphs was presented by Kochol [9], but is more complicated.

Dually chordal graphs are a subclass of clique-chordal graphs [2]. The next lemma will show, that a 3-colouring for clique-chordal graphs can be found with Algorithm 1.

Lemma 4. *Each block of a clique-chordal graphs is locally connected.*

Proof. Let $G = (V, E)$ be a clique-chordal graph and $K = (V_k, E_k)$ its clique graph. Assume that G has a block which is not locally connected. Then there is a vertex v such that its neighbourhood is not connected and it is no articulation point. Therefore, there is a chordless cycle \mathcal{C} fulfilling the conditions that there is no path in $G[N(v)]$ connecting the two neighbours of v in \mathcal{C} .

Let \mathcal{C}_G be the smallest \mathcal{C} and using the following notation: u, v, w and a, b, c are vertices of \mathcal{C}_G with $\{uv, vw, ab, cd\} \subseteq E$, $u \notin \{b, c\}$, $v \notin \{a, b, c\}$, $w \notin \{a, b\}$, $a \notin \{v, w\}$, $b \notin \{u, v, w\}$ and $c \notin \{u, v\}$. If \mathcal{C}_G is a C_4 , then $u = a$ and $w = c$. Figure 2 illustrates this notation.

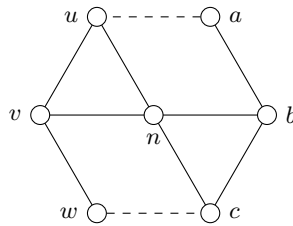


Fig. 2. The cycle \mathcal{C}_G and the vertex n .

There is no clique in G that contains two edges of \mathcal{C}_G , otherwise \mathcal{C}_G would not be chordless. Thus, each edge represents another clique of G . Let k_{uv} , k_{vw} ,

k_{ab} and k_{bc} denote these cliques. Because the cliques have a common vertex if the corresponding edges have one, they are forming a cycle \mathcal{C}_K in K .

K is chordal. Therefore, \mathcal{C}_K has at least one chord. Without loss of generality, let $k_{uv}k_{bc}$ be such a chord. Thus, k_{uv} and k_{bc} have a common vertex n that is adjacent to u , v , b and c .

If $w = c$, n is adjacent to w . If $w \neq c$, the sequence (v, n, c, \dots, w) induces a chordless cycle smaller than \mathcal{C}_G . In both cases \mathcal{C}_G is not the smallest cycle \mathcal{C} such that there is no path in $G[N(v)]$ connecting the two neighbours of v in \mathcal{C} .

Therefore, G is nearly locally connected. \square

It follows:

Corollary 3. *The 3-Colourability problem can be solved in linear time for dually chordal and clique chordal graphs.*

6 Conclusion

After introducing dually chordal graphs and investigating the connection between K_4 -free dually chordal and perfect graphs, this paper presented a linear time algorithm to find a 3-colouring by using the structure of locally connected graphs. This algorithm also computes a correct 3-colouring for each graph whose blocks are locally connected.

Acknowledgement. The author is grateful to H. N. de Ridder for stimulating discussions.

References

1. Brandstädt, A., Chepoi, V.D., Dragan, F.F.: The algorithmic use of hypertree structure and maximum neighbourhood orderings. *Discrete Applied Mathematics* 82(1–3), 43–77 (1998)
2. Brandstädt, A., Dragan, F., Chepoi, V., Voloshin, V.: Dually chordal graphs. *SIAM Journal on Discrete Mathematics* 11(3), 437–455 (1998)
3. Chudnovsky, M., Cornuéjols, G., Liu, X., Seymour, P., Vušković, K.: Recognizing Berge Graphs. *Combinatorica* 25, 143–186 (March 2005)
4. Chudnovsky, M., Robertson, N., Seymour, P., Thomas, R.: The strong perfect graph theorem. *Annals of Mathematics* 164, 51–229 (2006)
5. de Figueiredo, C.M., Meidanis, J., de Mello, C.P.: Total-chromatic number and chromatic index of dually chordal graphs. *Information Processing Letters* 70(3), 147–152 (1999)
6. Grötschel, M., Lovász, L., Schrijver, A.: Polynomial algorithms for perfect graphs. In: Berge, C., Chvátal, V. (eds.) *Topics on Perfect Graphs*, North-Holland Mathematics Studies, vol. 88, pp. 325–356. North-Holland (1984)
7. Guruswami, V., Khanna, S.: On the hardness of 4-coloring a 3-colorable graph. *SIAM Journal on Discrete Mathematics* 18(1), 30–40 (2004)
8. Karp, R.: Reducibility among combinatorial problems. In: Miller, R., Thatcher, J. (eds.) *Complexity of Computer Computations*, pp. 85–103. Plenum Press (1972)

9. Kochol, M.: Linear algorithm for 3-coloring of locally connected graphs. In: Jansen, K., Margraf, M., Mastrolilli, M., Rolim, J. (eds.) *Experimental and Efficient Algorithms*, pp. 191–194. No. 2647 in *Lecture Notes in Computer Science*, Springer Berlin / Heidelberg (2003)
10. Leitert, A.: *Das Dominating Induced Matching Problem für azyklische Hypergraphen*. Diploma thesis, University of Rostock, Germany (2012), in German
11. Stockmeyer, L.: Planar 3-colorability is polynomial complete. *SIGACT News* 5(3), 19–25 (Jul 1973)
12. Tarjan, R.E., Yannakakis, M.: Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs. *SIAM J. Comput.* 13, 566–579 (1984)